Worcester County Mathematics League

Varsity Meet 1 - October 9, 2024

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 1 - October 9, 2024
Answer Key

Round 1 - Arithmetic

1. $\frac{11}{4}$
2. -32
3. 8

Round 2 - Algebra I

1. $x^4 - 13x^2 + 36$
2. 12
3. $1 - \frac{\sqrt{6}}{3}$ or $\frac{3-\sqrt{6}}{3}$

Round 3 - Set Theory

1. $\{2, 4\}$
2. 8
3. 18

Round 4 - Measurement

1. 280
2. 450
3. $\frac{3\sqrt{3}}{4}$

Round 5 - Polynomial Equations

1. $(-4, -21)$ (exact order)
2. $\{a, -a, b\}$ (any order, need all three)
3. $\frac{b^2-1}{4}$ or $\frac{b^2}{4} - \frac{1}{4}$

Team Round

1. $(8, 39)$
2. 27
3. 54
4. $72\sqrt{3}$
5. -13
6. 10
7. $3x - 1$
8. $12 - 8\sqrt{2}$
9. -4

Worcester County Mathematics League
Varsity Meet 1 - October 9, 2024
Round 1 - Arithmetic

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Simplify the following expression. Express your answer as a rational number in the form $\frac{m}{n}$.

$$\frac{\left(2^{-2}+2^{-1}\right)\left(2^{-3}\right)+2^{-2}}{2^{-3}}$$

2. Let $a\Delta b=2a+b$ and $a\nabla b=a-2b$, and $7\Delta x=y\nabla 7$. Find:

$$(x-y)\,\nabla 2$$

3. There are exactly two different pairs of unit fractions that sum to $\frac{1}{2}$:

$$\begin{aligned}\frac{1}{2}&=\frac{1}{4}+\frac{1}{4}\\&=\frac{1}{3}+\frac{1}{6}\end{aligned}$$

where $\frac{1}{6}+\frac{1}{3}$ and $\frac{1}{3}+\frac{1}{6}$ are considered to be the same pair. Find the number of different pairs of unit fractions that sum to $\frac{1}{12}$.

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. ____________________

Douglas, Hopkinton, Athol/QSC

Worcester County Mathematics League
Varsity Meet 1 - October 9, 2024
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Express the following as a polynomial in standard form:

$$(x+2)(x-3)(x-2)(x+3)$$

2. Two pipes of different diameters fill a storage tank in 9 hours. The larger pipe fills the tanks three times faster than the smaller pipe. How many hours would it take to fill the tank if only the larger pipe were used?

3. What is the smallest number such that $\frac{1}{2}$ of it and $\frac{1}{6}$ of its reciprocal sum to 1?

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________ hours

(3 pts) 3. ____________________

Auburn, Bancroft, Algonquin

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Let $A \cap B$ be the intersection of sets A and B, U be the universal set and A^c be the complement of set A, that is, the elements of the universal set that are not members of A. If $U = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $A = \{0, 5, 6, 7\}$ and $B = \{0, 2, 4, 6\}$, find:

$$(A^c \cap B)$$

2. Set A has 252 fewer subsets than set B has. How many elements does set B have?

3. Set A has 46 elements, set B has 23 elements, set C has 55 elements, $A \cap B \cap C$ has 5 elements and $A \cup B \cup C$ has 96 elements. How many elements are in exactly two of the three sets?

ANSWERS

(1 pt) 1. $\{$ ____________________ $\}$

(2 pts) 2. ____________________

(3 pts) 3. ____________________

Worcester Academy, Bartlett, Bancroft

Worcester County Mathematics League
Varsity Meet 1 - October 9, 2024
Round 4 - Measurement

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Farmer Bob wants to fence in an 8000 square foot rectangular grazing area for his cows. The 80 foot long side of his barn will be one side of the rectangle. The other three sides of the rectangle will be bound by fencing. How many feet of fencing does Farmer Bob need to buy to fence in this area?

2. A cube with surface area 150cm^2 is sliced into thirds top to bottom, left to right, and front to back, creating 27 congruent cubes. What is the sum of the surface areas of those 27 cubes, measured in cm^2?

3. Alice has two solid spheres of different sizes. She slices the larger sphere in half, dividing it into two congruent solid hemispheres. She observes that the surface area of the second, smaller sphere is exactly equal to the surface area of one of the hemispheres. The volume of the smaller sphere is x times the volume of one of the hemispheres. Find x in simplest form.

ANSWERS

(1 pt) 1. ______________________ ft

(2 pts) 2. ______________________ cm^2

(3 pts) 3. ______________________

Burncoat, Shepherd Hill/QSC, AMSA Charter

Worcester County Mathematics League
Varsity Meet 1 - October 9, 2024
Round 5 - Polynomial Equations

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Polynomial $p(x) = x^2 + bx + c$ has roots -3 and 7. Find the ordered pair (b, c).

2. Solve the equation below for x. Express your answer in terms of constants a and b.

$$x^3 - bx^2 - a^2x + a^2b = 0$$

3. The roots of $x^2 + bx + c = 0$ differ by 1. Express c in as a polynomial in terms of b.

ANSWERS

(1 pt) 1. $(b, c) = \Big(\rule{4cm}{0.4pt}\Big)$

(2 pts) 2. $x \in \Big\{\rule{5cm}{0.4pt}\Big\}$

(3 pts) 3. $c =$ ______________________

Tahanto, Groton-Dunstable, Shepherd Hill

Worcester County Mathematics League
Varsity Meet 1 - October 9, 2024
Team Round

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. The expression below can be simplified to a ratio of integers $\frac{m}{n}$, where m and n have no common factors. Find the ordered pair (m, n).

$$\frac{\frac{2}{3}+\frac{1}{2}}{2-\frac{1}{4}} \div \frac{\frac{5}{6}+\frac{1}{4}}{\frac{1}{3}}$$

2. Marcy has a total of 100 dimes and quarters. The total value of the coins is \$14.05. How many quarters does Marcy have?

3. A group of students were asked whether they liked two ice cream flavors: strawberry and mocha chip. Of the group, 80% liked strawberry, 70% liked mocha chip, and 4% liked neither flavor. What percentage of students liked both flavors?

4. The lengths of the bases of an isosceles trapezoid are 14cm and 22cm. The measure of a base angle is $60°$. Find the exact area of the trapezoid in cm^2. Express your answer in simplest form.

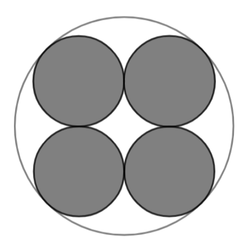
5. Given $p(x) = ax^2 + bx + c$, $p(1) = 10$, $p(2) = 9$, and $p(3) = 4$, find $a + 2b - 3c$.

6. Let $n = 20^4 \cdot 5^6$. How many digits does n have?

7. Simplify the following expression completely:

$$\frac{(3x+1)^2(3x-1)-(3x-1)^3}{12x}$$

8. A circle of radius 1 contains four congruent circles. Each of the four circles is tangent to the larger (radius 1) circle, and each is tangent to two other congruent circles, as shown in the figure at right. The sum of the areas of the four congruent circles is equal to πx. Find x.



9. When polynomial $p(x) = x^2 + 3x - 10$ is divided by $x - a$, the remainder is -6. When $p(x)$ is divided by $x - a + 1$, the remainder is 0. Find a.

Worcester County Mathematics League
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Team Round Answer Sheet

ANSWERS

1. $(m, n) = \big(\underline{\hspace{4cm}}\big)$

2. ______________________

3. ______________________ %

4. ______________________ cm^2

5. ______________________

6. ______________________

7. ______________________

8. ______________________

9. ______________________

Worcester County Mathematics League

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5. -13
6. 10
7. $3x - 1$
8. $12 - 8\sqrt{2}$
9. -4

Round 1 - Arithmetic

1. Simplify the following expression. Express your answer as a rational number in the form $\frac{n}{m}$:

$$\frac{\left(2^{-2}+2^{-1}\right)\left(2^{-3}\right)+2^{-2}}{2^{-3}}$$

Solution: Note that dividing the numerator by 2^{-3} is the same as multiplying the numerator by its reciprocal, 2^3. This fact is demonstrated by multiplying both the numerator and denominator by 2^3 and noting that the denominator is $2^3 \cdot 2^{-3} = 2^{3+(-3)} = 2^0 = 1$. Therefore:

$$\begin{aligned}\frac{\left(2^{-2}+2^{-1}\right)\left(2^{-3}\right)+2^{-2}}{2^{-3}} &= 2^3\left[\left(2^{-2}+2^{-1}\right)\left(2^{-3}\right)+2^{-2}\right] \\ &= 2^3 \cdot 2^{-3}\left(2^{-2}+2^{-1}\right)+2^3 \cdot 2^{-2} \\ &= 2^{3-3}\left(\frac{1}{4}+\frac{1}{2}\right)+2^{3-2} \\ &= 2^0 \cdot \frac{3}{4}+2^1 = \frac{3}{4}+2 = \frac{3}{4}+\frac{8}{4} = \boxed{\frac{11}{4}}.\end{aligned}$$

2. Let $a\Delta b = 2a+b$ and $a\nabla b = a-2b$, and $7\Delta x = y\nabla 7$. Find:

$$(x-y)\,\nabla 2$$

Solution: Start with the last given equation, and apply the given definitions for Δ and ∇:

$$\begin{aligned}7\Delta x &= y\nabla 7 \\ 2\cdot 7+x &= y-2\cdot 7 \\ 14+x &= y-14\end{aligned}$$

Next, subtract 14 and y from each side of the equation so that $x-y$ is isolated on the left side of the equation:

$$x-y = -14-14 = -28$$

Now substitute -28 for $x-y$ and apply the definition for ∇: $(x-y)\,\nabla 2 = -28\nabla 2 = -28-2(2) = -28-4 = \boxed{-32}$.

3. There are exactly two different pairs of unit fractions that sum to $\frac{1}{2}$:

$$\begin{aligned}\frac{1}{2} &= \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{3} + \frac{1}{6}\end{aligned}$$

where $\frac{1}{6} + \frac{1}{3}$ and $\frac{1}{3} + \frac{1}{6}$ are considered to be the same pair. Find the number of different pairs of unit fractions that sum to $\frac{1}{12}$.

Solution: Start by noting that one sixth of the two pairs of fractions that sum to $\frac{1}{2}$ will sum to $\frac{1}{12}$ and in particular, $\frac{1}{12} = \frac{1}{24} + \frac{1}{24}$. Now let $\frac{1}{m} + \frac{1}{n} = \frac{1}{12}$ where m and n are positive integers and $n \leq m$, so that $\frac{1}{n} \geq \frac{1}{m}$. Note that values of n are limited: because $\frac{1}{n} < \frac{1}{12}$, $n > 12$; because $\frac{1}{n} \geq \frac{1}{m}$, $n \leq 24$. Thus $12 < n \leq 24$ and n can take on 12 values from 13 to 24. Now

$$\frac{1}{12} - \frac{1}{n} = \frac{1}{m}.$$

Combine the two fractions on the left, then multiply both sides of the equation by m and $12n$ and divide both sides by $n - 12$ to create an expression for m for any value of n:

$$\begin{aligned}\frac{n-12}{12n} &= \frac{1}{m} \\ m &= \frac{12n}{n-12}\end{aligned}$$

Thus, for m to be an integer, $n - 12$ must be a factor of $12n$.

Proceed by testing the 12 possible values of n. For $n = 13$, $m = 12 \cdot 13 = 156$; 13 works because $13 - 12 = 1$ is a factor of $12 \cdot 13$. For $n = 14$, $m = 12\frac{14}{2} = 12 \cdot 7 = 84$ 14 works because 2 is a factor of $12 \cdot 14$. Similarly, $n = 15$ and $n = 16$ work, but not $n = 17$: $17 - 12 = 5$ does not divide $12n = 12 \cdot 17$. Proceeding, $n = 18, 20, 21$ and 24 work, but not $n = 19, 22$ or 23. In total, 4 of the twelve integer values from 13 to 24 don't work, and there are $\boxed{8}$ possible pairs of unit fractions that sum to $\frac{1}{12}$.

Round 2 - Algebra I

1. Express the following as a polynomial in standard form:

$$(x+2)(x-3)(x-2)(x+3)$$

Solution: This expression is most easily simplified by applying the difference of squares identity: $a^2 - b^2 = (a+b)(a-b)$. Reorder the product terms to reveal that the right hand side of this identity appears twice, once for $b = 2$ and once for $b = 3$. Proceeding:

$$\begin{aligned}(x+2)(x-3)(x-2)(x+3) &= (x+2)(x-2)(x+3)(x-3)\\ &= \left(x^2-2^2\right)\left(x^2-3^2\right) = \left(x^2-4\right)\left(x^2-9\right)\end{aligned}$$

Next multiply the two polynomials using the distributive property, group terms, and write the polynomial with monomials in order, highest exponent to lowest:

$$\begin{aligned}\left(x^2-4\right)\left(x^2-9\right) &= x^2\left(x^2-9\right) - 4\left(x^2-9\right)\\ &= x^2 \cdot x^2 - 9x^2 - 4x^2 - 4(-9) = x^{2+2} - (4+9)x^2 + (-1)^2\, 4 \cdot 9\\ &= \boxed{x^4 - 13x^2 + 36}\end{aligned}$$

2. Two pipes of different diameters fill a storage tank in 9 hours. The larger pipe fills the tanks three times faster than the smaller pipe. How many hours would it take to fill the tank if only the larger pipe were used?

Solution: Let h be the number of hours that that it takes to fill the tank using only the larger pipe. The larger pipe therefore fills at the rate of $\frac{1}{h}$ tanks per hour. Then the smaller pipe will take $3h$ hours to fill the tank by itself, and it fills $\frac{1}{3h}$ tanks per hour. The two pipes together will fill tanks at the rate of:

$$\begin{aligned}\frac{1}{h} + \frac{1}{3h} &= \frac{1}{h}\left(1 + \frac{1}{3}\right) = \frac{1}{h}\frac{4}{3}\\ &= \frac{4}{3h}\end{aligned}$$

tanks per hour. Therefore they will take $\frac{3h}{4}$ hours to fill one tank. Set this expression equal to 9 and solve for h:

$$\begin{aligned}\frac{3h}{4} &= 9\\ h &= \frac{4}{3} \cdot 9 = 4 \cdot \frac{9}{3} = 4 \cdot 3 = \boxed{12}\end{aligned}$$

3. What is the smallest number such that $\frac{1}{2}$ of it and $\frac{1}{6}$ of its reciprocal sum to 1?

Solution: Let x be the number that is asked for. Then $\frac{x}{2} + \frac{1}{6} \cdot \frac{1}{x} = 1$. Multiply both sides of this equation by the common denominator $6x$ and solve for x:

$$6x\left(\frac{x}{2} + \frac{1}{6} \cdot \frac{1}{x}\right) = 6x \cdot 1$$
$$\frac{6x^2}{2} + \frac{6x}{6x} = 6x$$
$$3x^2 + 1 = 6x$$

The equivalent quadratic equation, $3x^2 - 6x + 1 = 0$, can be solved using the quadratic formula:

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4 \cdot}}{2 \cdot 3} = \frac{6 \pm \sqrt{36 - 12}}{6} = \frac{6}{6} \pm \frac{\sqrt{24}}{6} = 1 \pm \frac{2\sqrt{6}}{6} = 1 \pm \frac{\sqrt{6}}{3}$$

Then the smaller of these two solutions is $\boxed{1 - \frac{\sqrt{6}}{3}}$.

Round 3 - Set Theory

1. Let $A \cap B$ be the intersection of sets A and B, U be the universal set and A^c be the complement of set A, that is, the elements of the universal set that are not members of A. If $U = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $A = \{0, 5, 6, 7\}$ and $B = \{0, 2, 4, 6\}$, find:

$$(A^c \cap B)$$

Solution: Apply the definition of a complement:

$$A^c = \{1, 2, 3, 4\}$$

Recall that the intersection of two sets is the set of all elements common to the two sets. Then $A^c \cap B = \boxed{\{2, 4\}}$.

2. Set A has 252 fewer subsets than set B has. How many elements does set B have?

Solution: Recall that a set with k elements has 2^k subsets, including the empty set and the subset itself. This fact is explained as follows. For a given subset, there are two possibilities for each element: in the subset, or not in the subset. The choices for the k elements are independent. The number of subsets is therefore the product of k 2's, or 2^k.

Let A have m elements and B have n elements. Then B has $2^n - 2^m = 2^m\left(2^{n-m} - 1\right) = 252$ more subsets than A. Note that 252 factors into a product of a power of two and one less than a power of two: $252 = 4 \cdot 63 = 4\,(64 - 1) = 2^2\left(2^6 - 1\right)$. Then $2^m\left(2^{n-m} - 1\right) = 2^2\left(2^6 - 1\right)$ so that $m = 2$ and $n - m = 6$, or $n = 6 + m = 6 + 2 = \boxed{8}$.

Alternative Solution: Note that set B must have more than 252 subsets, since the number of subsets of A is positive, and it must be a power of 2. The powers of 2 are $2, 4, 8, 16, 32, 64, 128, 256, 512, \ldots$, found by doubling each number in succession. The first number in this list that is greater than 252 is 256, which is the eighth number, so that $256 = 2^8$. Then set A has $256 - 252 = 4 = 2^2$ subsets, and A has 2 elements. Therefore B has $\boxed{8}$ elements.

Note that B cannot have 9 elements because set A would have $2^9 - 252 = 512 - 252 = 260$ subsets, and 260 is not a power of 2. Moreover, if n is the number of elements of B and $n > 8$ then $n - 252$ will be between 2^n and the next lower number of elements, that is, $2^{n-1} < 2^n - 252 < 2^n$ for $n > 8$. This fact is true because the difference between 2^n and 2^{n-1} is 256 for $n = 9$ and this difference increases for larger values of n.

3. Set A has 46 elements, set B has 23 elements, set C has 55 elements, $A \cap B \cap C$ has 5 elements and $A \cup B \cup C$ has 96 elements. How many elements are in exactly two of the three sets?

Solution: Recall the Principle of Inclusion-Exclusion for three sets A, B, and C:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

where $|A|$ is the number of elements in set A. Now let $y = |A \cap B| + |B \cap C| + |A \cap C|$, the sum of the counts of the three pairwise intersection. Substitute y, as well as the values given in the problem statement ($|A| = 46$, $|B| = 23$, $|C| = 55$, $|A \cap B \cap C| = 5$, $|A \cup B \cup C| = 96$) into this equation and solve for y:

$$\begin{aligned} 96 &= 46 + 23 + 55 - (|A \cap B| + |B \cap C| + |A \cap C|) + 5 \\ &= 46 + 23 + 55 + 5 - y \\ &= 129 - y \\ y &= 129 - 96 = 33 \end{aligned}$$

Let x be the number of elements in exactly two sets. Note that y overcounts x because each intersection of two sets includes 5 elements that are present in all three sets. Therefore $y = x + 3 \cdot |A \cap B \cap C| = x + 3 \cdot 5 = x + 15$, or $x = y - 15$. Substitute $y = 33$ and $x = 33 - 15 = \boxed{18}$.

Round 4 - Measurement

1. Farmer Bob wants to fence in an 8000 square foot rectangular grazing area for his cows. The 80 foot long side of his barn will be one side of the rectangle. The other three sides of the rectangle will be bound by fencing. How many feet of fencing does Farmer Bob need to buy to fence in this area?

Solution: Recall the area of a rectangle with side lengths l (for length) and w (for width) is equal to $l \cdot w$. Then if $l = 80$ and the area $A = 8000$, $w = A \div l = 8000 \div 80 = 100$. Farmer Bob needs to buy enough fencing for the two 100 foot sides of the rectangle and one 80 foot side of the rectangle. Thus, Farmer Bob needs to buy $2 \cdot 100 + 1 \cdot 80 = 200 + 80 = \boxed{280}$ feet of fencing.

2. A cube with surface area 150cm^2 is sliced into thirds top to bottom, left to right, and front to back, creating 27 congruent cubes. What is the sum of the surface areas of those 27 cubes, measured in cm^2?

Solution: Recall the terminology of a cube: the corners are called *vertices*, the square sides are called *faces*, and the line segments, each connecting two vertices, that form the sides of the faces and are called *edges*. After slicing, there will be 27 cubes, each of which has a edge length equal to $\frac{1}{3}$ the edge length of the original cube. The area of each face will then be $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ of the area of one of the faces of the original cube. More to the point, the surface area (6 times the area of one face) of one of the 27 cubes will be $\frac{1}{9}$ of the surface area of the original, larger cube. The sum of the surface areas of the 27 cubes will then be $27\left(\frac{1}{9}\right) = 3$ times the surface area of the original cube. Therefore this sum is equal to $3 \cdot 150 = \boxed{450}$cm^2

3. Alice has two solid spheres of different sizes. She slices the larger sphere in half, dividing it into two congruent solid hemispheres. She observes that the surface area of the second, smaller sphere is exactly equal to the surface area of one of the hemispheres. The volume of the smaller sphere is x times the volume of one of the hemispheres. Find x in simplest form.

Solution: Recall that the surface area of a sphere is $4\pi r^2$, where r is the radius of the sphere. The surface area of a hemisphere is equal to half the surface area of a sphere plus the area of the circular surface where the sphere was sliced, or $\frac{4\pi r^2}{2} + \pi r^2 = 2\pi r^2 + \pi r^2 = 3\pi r^2$. Let r_1 be the radius of the larger sphere, also the radius of the hemispheres, and let r_2 be the radius of the smaller sphere. Then $4\pi r_2^2 = 3\pi r_1^2$. Solve for the ratio $\frac{r_2}{r_1}$:

$$4\pi r_2^2 = 3\pi r_1^2$$
$$\frac{r_2^2}{r_1^2} = \frac{3\pi}{4\pi} = \frac{3}{4}$$
$$\frac{r_2}{r_1} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Recall that the volume of a sphere with readius r is $\frac{4}{3}\pi r^3$. The volume of a hemisphere is half that: $\frac{2}{3}\pi r^3$. Therefore the ratio of the volume of the smaller sphere to the volume of the hemisphere is:

$$\frac{\frac{4}{3}\pi r_2^3}{\frac{2}{3}\pi r_1^3} = \frac{2r_2^3}{r_1^3} = 2\left(\frac{r_2}{r_1}\right)^3 = 2\left(\frac{\sqrt{3}}{2}\right)^3 = 2\frac{(\sqrt{3})^3}{2^3} = 2\frac{3\sqrt{3}}{8} = \boxed{\frac{3\sqrt{3}}{4}}$$

Round 5 - Polynomial Equations

1. Polynomial $p(x) = x^2 + bx + c$ has roots -3 and 7. Find the ordered pair (b, c).

Solution: Recall that r is a root of polynomial $p(x)$ if $p(r) = 0$. Then $p(-3) = (-3)^2 + b(-3) + c = 0$, and $p(7) = 7^2 + 7b + c = 0$. They can be written as $-3b + c + 9 = 0$ and $7b + c + 49 = 0$: two equations in two unknowns:

$$-3b + c = -9$$
$$7b + c = -49$$

Subtract the top equation from the second equation to yield $10b = -40$, and $b = -4$. Then substitute $b = -4$ into the top equation: $-3(-4) + c = -9$, or $c + 12 = -9$, and $c = -21$. Therefore $(b, c) = \boxed{(-4, -21)}$

Alternative solution: Vieta's formula for a quadratic equation of the form $ax^2 + bx + c = 0$ says that the sum of the roots of this equation are equal to $-\frac{b}{a}$ and the product of the roots is equal to $\frac{c}{a}$. In this case, $a = 1$ and the sum of the roots is $-3 + 7 = 4 = -b$. Therefore -$\frac{b}{a} = -b = 4$ and $b = -4$.

Now the product of the roots is $(-3)(7) = -21 = \frac{c}{a} = c$ because $a = 1$. Therefore $(b, c) = \boxed{(-4, -21)}$.

2. Solve the equation below for x. Express your answer in terms of constants a and b.

$$x^3 - bx^2 - a^2x + a^2b = 0$$

Solution: Note that the first two terms have a common factor of x^2, the last two terms have a common factor of a^2, and that this polynomial can be factored:

$$\begin{aligned} x^3 - bx^2 - a^2x + a^2b &= x^2(x - b) + a^2(-x + b) \\ &= x^2(x - b) - a^2(x - b) \\ &= \left(x^2 - a^2\right)(x - b) = 0 \end{aligned}$$

Therefore either $(x^2 - a^2) = 0$ or $x - b = 0$. In the first case, $(x - a)(x + a) = 0$, so $x = a$ or $x = -a$. In the second case, $x = b$. The solution set for x = $\boxed{\{a, -a, b\}}$.

3. The roots of $x^2 + bx + c = 0$ differ by 1. Express c as a polynomial in terms of b.

Solution: Recall that quadratic formula states that the two roots to the quadratic equation $ax^2 + bx + c = 0$ are:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For this problem, $a = 1$ and the two roots are $r_1 = \frac{-b+\sqrt{b^2-4c}}{2}$ and $r_2 = \frac{-b-\sqrt{b^2-4c}}{2}$. Now set $r_1 - r_2 = 1$:

$$\begin{aligned} r_1 - r_2 &= \frac{-b + \sqrt{b^2 - 4c}}{2} - \left(\frac{-b - \sqrt{b^2 - 4c}}{2}\right) \\ &= \frac{-b + \sqrt{b^2 - 4c} - (-b) - (-\sqrt{b^2 - 4c})}{2} \\ &= \frac{-b + b + \sqrt{b^2 - 4c} + \sqrt{b^2 - 4c}}{2} \\ &= \frac{2\sqrt{b^2 - 4c}}{2} = \sqrt{b^2 - 4c} = 1 \end{aligned}$$

Now square both sides of the last equation, and $b^2 - 4c = 1$. Add $4c - 1$ to both sides and $4c = b^2 - 1$. Then $c = \boxed{\frac{b^2 - 1}{4}}$.

Team Round

1. The expression below can be simplified to a ratio of integers $\frac{m}{n}$, where m and n have no common factors. Find the ordered pair (m, n).

$$\frac{\frac{2}{3}+\frac{1}{2}}{2-\frac{1}{4}} \div \frac{\frac{5}{6}+\frac{1}{4}}{\frac{1}{3}}$$

Solution: Begin by combining fractions by finding the common denominator:

$$\frac{\frac{2}{3}+\frac{1}{2}}{2-\frac{1}{4}} \div \frac{\frac{5}{6}+\frac{1}{4}}{\frac{1}{3}} = \frac{\frac{4}{6}+\frac{3}{6}}{\frac{8}{4}-\frac{1}{4}} \div \frac{\frac{10}{12}+\frac{3}{12}}{\frac{1}{3}}$$
$$= \frac{\frac{7}{6}}{\frac{7}{4}} \div \frac{\frac{13}{12}}{\frac{1}{3}}$$

Recall that dividing by a fraction is the same as multiplying by the reciprocal of the fraction, and:

$$\frac{\frac{7}{6}}{\frac{7}{4}} \div \frac{\frac{13}{12}}{\frac{1}{3}} = \left(\frac{4}{7}\cdot\frac{7}{6}\right) \div \left(3\cdot\frac{13}{12}\right)$$
$$= \frac{2}{3} \div \frac{13}{4}$$
$$= \frac{2}{3}\cdot\frac{4}{13} = \frac{8}{39}$$

and $(m, n) = \boxed{(8, 39)}$.

2. Marcy has a total of 100 dimes and quarters. The total value of the coins is \$14.05. How many quarters does Marcy have?

Solution: One way to solve this problem is to set up a system of two equations in two unknowns. Let q be the number of quarters and d be the number of dimes. Since dimes have a value of 10 cents, quarters have a value of 25 cents, and the total value of the coins is 1405 cents:

$$d + q = 100$$
$$10d + 25q = 1405.$$

Subtract 10 times the top equation from the bottom equation and $15q = 405$, or $q = 405 \div 15 = \boxed{27}$.

Alternative Solution: If all 100 coins were dimes, they would have a value of 10.00, or 1000 cents. Each time a quarter is substituted for a dime, the value of the coins is increased by $25 - 10 = 15$ cents. Enough quarters need to be substituted for dimes to increase the value by \$4.05 or $405 \div 15 = 27$, as before.

3. A group of students were asked whether they liked two ice cream flavors: strawberry and mocha chip. Of the group, 80% liked strawberry, 70% liked mocha chip, and 4% liked neither flavor. What percentage of students liked both flavors?

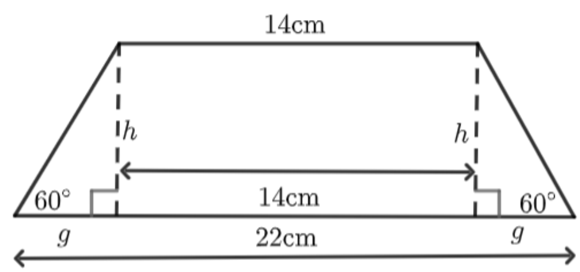
Solution: If 4% liked neither flavor, then $100 - 4 = 96\%$ like at least one flavor. Apply the principle of inclusion-exclusion for two sets S (students who like strawberry) and C(students who like mochachip):

$$\begin{aligned}|S \cup C| = 96\% &= |S| + |C| - |S \cap C| \\ &= 80\% + 70\% - x\end{aligned}$$

where we let x be the percentage of students who liked both flavors. Solving this equation for x, we have $x = 80\% + 70\% - 96\% = 150\% - 96\% = \boxed{54}\%$.

4. The lengths of the bases of an isosceles trapezoid are 14cm and 22cm. The measure of a base angle is 60°. Find the exact area of the trapezoid in cm^2. Express your answer in simplest form.

Solution: First draw and label a figure. Draw altitudes from the end of the short base to the longer base, shown at right as dotted lines. Because the trapezoid is isosceles, its base angles are congruent. The altitudes therefore divide the trapezoid into two 30°-60°-90° triangles and a rectangle of length 14cm and height h, as yet unknown.



Solution: (continued) Let h be the length of the altitude and g be the length of the shorter leg of the triangle. Now the longer base is composed of the two short legs of the triangles and the length of the rectangle. Therefore $14 + 2g = 22$, $2g = 22 - 14 = 8$ and $g = 4$. Recall that the sides of a 30°-60°-90° triangle are in proportion $1 : \sqrt{3} : 2$. Therefore $h = g\sqrt{3} = 4\sqrt{3}$. Recall that the area of a trapezoid is equal to its altitude times the average of its base lengths. Therefore the area of this trapezoid is $4\sqrt{3}\left(\frac{14+22}{2}\right) = 4\sqrt{3} \cdot 18 = \boxed{72\sqrt{3}}$cm^2.

5. Given $p(x) = ax^2 + bx + c$, $p(1) = 10$, $p(2) = 9$, and $p(3) = 4$, find $a + 2b - 3c$.

Solution: Substitute $x = 1$, $x = 2$, and $x = 3$ to create three equations in unknowns (a, b, c):

$$a(1)^2 + b(1) + c = a + b + c = 10$$
$$a(2)^2 + b(2) + c = 4a + 2b + c = 9$$
$$a(3)^2 + b(3) + c = 9a + 3b + c = 4$$

Proceed using the method of elimination. Subtract the top equation from the second equation and subtract the second equation from the third equation:

$$\begin{array}{lcl} 4a + 2b + c = 9 & \quad & 9a + 3b + c = 4 \\ -(a + b + c = 10) & & -\ (4a + 2b + c = 9) \\ \hline 3a + b \qquad = -1 & & 5a + b \qquad = -5 \end{array}$$

The third variable c has been eliminated leaving two equations in two unknowns. Subtract the left bottom equation from the right bottom equation, $(5a + b = -5) - (3a + b = -1)$, and $2a = -4$, or $a = -2$. Substitute $a = -2$ in the bottom left equation above and $3(-2)+b = -1$, or $b = -1+6 = 5$. Next, substitute these values for a and b into the first original equation $a + b + c = 10$ and $-2+5+c = 10$, or $c = 10+2-5 = 7$. Finally, $a+2b-3c = -2+2(5)-3(7) = -2+10-21 = \boxed{-13}$.

6. Let $n = 20^4 \cdot 5^6$. How many digits does n have?

Solution: First rewrite n in its prime factorization form, applying laws of exponents:

$$n = 20^4 \cdot 5^6 = \left(2^2 \cdot 5\right)^4 \cdot 5^6 = \left(2^2\right)^4 \cdot 5^4 \cdot 5^6 = 2^{2\cdot 4} \cdot 5^{4+6} = 2^8 \cdot 5^{10}$$

Note that $2^n \cdot 5^n = (2 \cdot 5)^n = 10^n$, and that $5^{10} = 5^{8+2} = 5^8 \cdot 5^2$. Then

$$n = 2^8 \cdot 5^{10} = 2^8 \cdot 5^8 \cdot 5^2 = (2 \cdot 5)^8 \cdot 5^2 = 10^8 \cdot 25$$

Now 10^8 is written as one with 8 zeros, so $n = 25 \cdot 10^8$ is written as 25 with 8 zeros, and n has $\boxed{10}$ digits.

7. Simplify the following expression completely:

$$\frac{(3x+1)^2(3x-1)-(3x-1)^3}{12x}$$

Solution: Let $n(x)$ be the numerator of the expression and note that $3x-1$ is a factor:

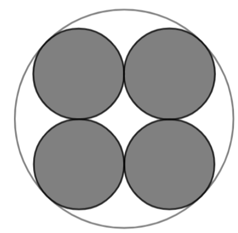
$$n(x) = (3x+1)^2(3x-1)-(3x-1)^3 = (3x-1)\left[(3x+1)^2-(3x-1)^2\right]$$

Next, note that the bracketed expression is a difference of squares and apply the identity $a^2-b^2 = (a+b)(a-b)$ with $a = 3x+1$ and $b = 3x-1$:

$$\begin{aligned} n(x) = (3x-1)\left[(3x+1)^2-(3x-1)^2\right] &= (3x-1)(3x+1+3x-1)(3x+1-(3x-1)) \\ &= (3x-1)(3x+3x+1-1)(3x+1-3x+1) \\ &= (3x-1)(6x)\cdot 2 = (3x-1)\cdot 12x \end{aligned}$$

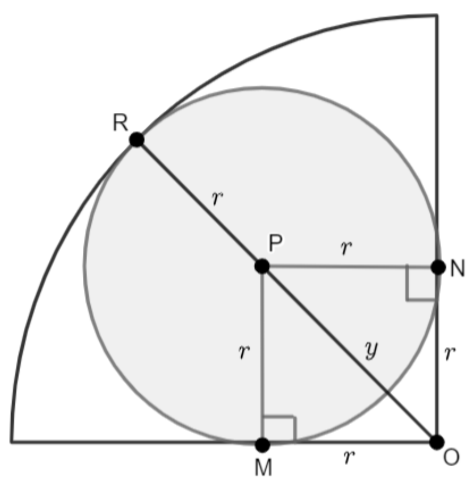
Now simplify the original expression using this identity for $n(x)$: $\frac{n(x)}{12x} = \frac{12x(3x-1)}{12x} = \boxed{3x-1}$.

8. A circle of radius 1 contains four congruent circles. Each of the four circles is tangent to the larger (radius 1) circle, and each is tangent to two other congruent circles, as shown in the figure at right. The sum of the areas of the four congruent circles is equal to πx. Find x.



Solution: Let r be the radius of one of the four shaded circles. Then the area of one of those circles is πr^2, and the total shaded area, the area of the four congruent circles, is $4\pi r^2$. It remains to find r.

Draw one vertical diameter and one horizontal diameter, each tangent to two shaded circles, to divide the larger circle into quarter circles. Each quarter circle is a 90° sector with an inscribed shaded circle. Draw one sector as shown at right with the center labeled O. Note that the shaded circle is tangent to both radii at M and N and tangent to the arc of the sector at R. Draw $\overline{OR}$, with $OR = 1$ because it is a radius of the larger circle. Note that it passes through P, the center of the shaded circle, so that $\overline{PR}$ is a radius of the shaded circle and $PR = r$. Draw the two radii (of the shaded circle) $\overline{PN}$ and $\overline{PM}$ as shown. Note that $NOMP$ is a square, that $PM = OM = r$, and $\overline{OP}$ is a diagonal. Then ΔOMP is an isosceles right triangle and $OP = y = r\sqrt{2}$.



Now $OR = OP + PR = r\sqrt{2} + r = 1$.

8. Solution (continued): Solve for r:

$$r\left(\sqrt{2}+1\right)=1$$
$$r=\frac{1}{\sqrt{2}+1}$$
$$=\frac{1}{\sqrt{2}+1}\frac{\sqrt{2}-1}{\sqrt{2}-1}=\frac{\sqrt{2}-1}{\left(\sqrt{2}\right)^2-1}=\frac{\sqrt{2}-1}{2-1}=\sqrt{2}-1$$

multiplying the rational expression top and bottom by the conjugate of the denominator and applying the difference of squares identity. Now the area of the shaded circle is

$$4\pi r^2=4\pi\left(\sqrt{2}-1\right)^2=4\pi\left(\left[\sqrt{2}\right]^2-2\sqrt{2}+1\right)$$
$$=4\pi\left(2-2\sqrt{2}+1\right)=4\pi\left(3-2\sqrt{2}\right)=\pi\left(\boxed{12-8\sqrt{2}}\right)$$

9. When polynomial $p(x)=x^2+3x-10$ is divided by $x-a$, the remainder is -6. When $p(x)$ is divided by $x-a+1$, the remainder is 0. Find a.

Solution: Recall the Remainder Theorem: when a polynomial $p(x)$ is divided by $x-b$, the remainder is equal to $p(b)$. In this case, set $p(a)$ to -6, the remainder: $p(a)=a^2+3a-10=-6$, which put into standard form is $a^2+3a-4=0$. This polynomial factors: $a^2+3a-4=(a+4)(a-1)=0$, so $a=-4$ or $a=1$.

Next, note that if $p(x)\div(x-b)$ leaves a remainder of zero then $x-b$ is a factor of $p(x)$, and recall the Factor Theorem: $x-b$ is a factor of if $p(x)$ if $p(b)=0$ and vice versa. In this case, $x-a+1$ divides $p(x)$ and is a factor, so $p(-(-a+1))=p(a-1)=0$, and $a-1$ is a root of $p(x)$. One way to proceed is to solve $p(a-1)=0$, but it is quicker to simply factor $p(x)$:

$$p(x)=x^2+3x-10=(x-2)(x+5)$$

and the roots of $p(x)$ are 2 and -5. If $a=1$, then $a-1=0$, which is not a root If $a=-4$, then $a-1=-5$, which is a root. The answer is therefore $a=\boxed{-4}$.